Port-based models for real-time supervision and diagnosis during oilwell acid stimulations

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Abstract. It is discussed the rigorous modeling of an oil wellbore during an acid stimulation with the purpose of predicting and monitoring the remotion of clogs on the formation around their bottom. The rigorous modeling of the fluidic system is approached as a composition of energy dissipating lumped subsystems interconnected by power conserving variables in a similar fashion considered in [13]. The resulting chain of lumped models serves to predict the bottomhole pressure during injection, simplifying the computational implementation of a system for real-time monitoring, prediction and diagnosis of acid stimulation operations.

1 Introduction

These days, preserving the productivity of existent oilwells is the first concern in the mexican oil industry.
A common maintenance operation to keep oilwell productivity is known in the oil industry as *matrix acid stimulation*. The purpose of a matrix acidizing treatment is to increase the productivity of an oil well by injecting a system of dissolving acids into the wellbore, and with this, dissolving the *near-wellbore formation damage*, commonly called *skin*. These acids tend to create pathways (*wormholes*) around the borehole that increase productivity. Similarly to *hydraulic fracturing*, the stimulation acids are necessarily pumped at pressures above the reservoir static pressure but in a range of pressures always below the reservoir fracture pressure, resulting in a cheaper operation, [3].
This operation is not always sucessful. Two determinant factors of success are the correct selection of fluids and additives and the operator practice in the field. While the selection of fluids is nowadays determined by the software developed by the mayor service distributors, the field practice can improved significantly by real-time monitoring of their operations.
Real-time monitoring of the performance of matrix acidizing treatments needs of a dynamic model that provides an *estimated bottomhole pressure during injection* P_{wfi} in order to diagnose the reduction of skin. This paper focuses on the modeling issues associated to the prediction of the bottomhole pressure. The complementary paper [12] concentrates on the algorithms for prediction of the skin factor along with the real-time computer implementation of the overall system.
Typically, an estimulation consist of the injection of a system of fluids on a predesigned

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schedule of dosaged stages, *e.g.* the injection of a ***preflush fluid*** (with dilluted hydrochloric acid) followed by several *main acids* (regular mud acid, a mixture of HCl and hydrofluoric HF acids), several ***overflushes*** (with more dilluted hydrochloric acid) and probably a *diverter slug* (very dilluted HCl), see Figure 1. After the necessary num-

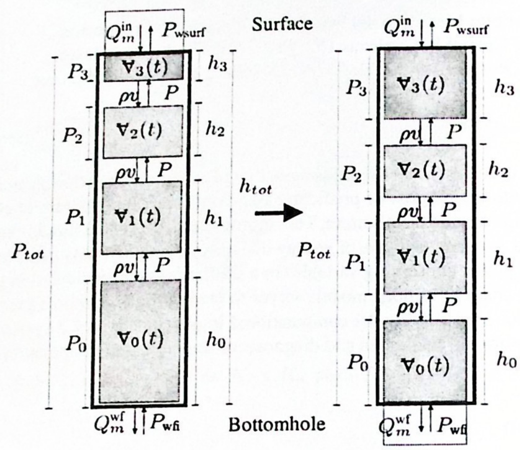

Fig. 1. Staggered energy transfer during stimulation

ber of these stages, the process is finished with a ***tubing displacement fluid*** (probably NH_4Cl brine or with foam), [3, 9].
The traditional approach of modeling this process consist on assuming the sequential stages of fluids to define a hydraulic column with an associated pressure drop on each stage, see *e.g.* [18, 15, 1, 23]. Nevertheless, the modeling difficulty of this process is mainly due to the different properties of the system of fluids during the sequential injection, along with the uncertainty introduced by each compressible stage.
In our approach, the resulting model consist of a system of differential-algebraic equations (DAEs) that keep track of the behavior of the control volumes of each fluid as they are introduced along the production tubing (P.T.) until the control volume integrates into the formation. Based on their properties it is possible to estimate the pressure drop of each stage and along with the information of the surface pressure, it is possible to provide an estimate of the bottom hole pressure during this injection. Though, everytime that a new fluid is added to the fluid column and everytime that the fluid at the bottom hole integrates to the formation, the column pressure calculation is being modified, see Fig. 1.
The main contribution of this work is a preliminar system-theoretic oriented, lumped modelling of the acid stimulation process. In particular, influenced by the port-based modelling of fluidic process systems presented in [13], we provide a differential-algebraic

model to predict in real-time the bottomhole pressure during injection, which is fundamental in order to predict the remotion of clogs due to formation damage. Based on a singular-pertubation type argument, the full fluid dynamic equations are reduced to DAEs consisting of a dynamic conservation equation restricted by an integral invariance principle on the momentum equation. With the use of an optimization-based algorithm to predict the pressure drop due to foams introduced in this paper, we formalize a method that guarantees satisfaction of the momentum equation in contrast to the iterative method by [6].

The paper is organized as follows. In Sec. 2 we provide the fundamental concepts of lumped modeling for staggered fluids using the dissipative system approach presented in [13]. In Sec. 3 we present the singular perturbation-based model reduction resulting in an optimization approach to satisfy the spatial restrictions of the problem during the calculations of pressure. In Sec. 4 we deal with the accommodation of volumes inside the wellbore in order to predict the bottomhole pressure. Throughout the paper we discuss the advantages of using the systemic modelling approach presented. Finally we draw some conclusions.

2 Port-based modeling of staggered fluids

An attractive paradigm for systemic modeling and interconnection of petroleum production systems are port-based models. In this modeling paradigm for control systems every subsystem is conceptualized as an energy distribution device whose exogenous (input-output) variables are power conserving. In particular, the influential paper [20] contributes to this viewpoint by including the (Euler) ideal fluid equations in the port-Hamiltonian formalism. This systemic formalism has several advantages in applications due their remarkable properties of energy preservation and closedness under system interconnection. This approach has been succesfully applied for uncompressible fluidic models like in open channels, see [7]. In the case of compressible fluids, in [13] the authors provided some *approximations* of distributed port-Hamiltonian equations for these fluids. The complete (exact) description of compressible fluids for non-ideal fluid dynamics in the port-Hamiltonian formalism still seems to be too difficult, see [14, 5] for recent progress of this formalism for irreversible Thermodynamics.

In the problem of this paper we deal with uncompressible, compressible, two-phase and non-newtonian fluids with complex reological behavior, and therefore a complete port-Hamiltonian should not be expected here. Nevertheless, the interconnection of subsystems is highly benefited by the overall systemic approach based on energy preserving interconnections. Moreover, in the distributed port-Hamiltonian formalism an *Eulerian frame* is assumed and in the problem of this paper it is sometimes convenient to fix a frame to the control volume $\forall$, resulting in a *Lagrangean frame*.

The modeling begins by identifying the functionals that characterize the properties of the fluid involved, see Table 1 in [13]. In particular there are *supply-rate densities* given by $r(x(t)) = \rho$ for the mass and $r(x(t)) = \rho v$ for the momentum with associated *supply-rate functionals* $R[r, \mathrm{v}, A] = -\int_A \rho\, \mathrm{v}\cdot\mathrm{n}\, dA$ and $R[r, \mathrm{v}, A] = -\int_A \rho v\, \mathrm{v}\cdot\mathrm{n}\, dA$ respectively. Furthermore, each control volume has mass and momentum *storage functionals* $S(x(t))$ denoted as $m = \int_\forall \rho\, d\forall$ and $M = \int_\forall \rho v\, d\forall$ respectively.

At the begining of the injection process, there is a remanent fluid inside the wellbore that defines the initial bottomhole wellbore pressure. As the first injection fluid from the (surface) top side displaces the remanent fluid to the botomhole side, such fluid reaches the bottomhole and it incorporates to the formation around the wellbore. It is therefore convenient to assume an Eulerian Frame attached to both extremes of the tubing. By mass conservation and the Transport Theorem,

$$\frac{dm_i}{dt} = \int_{\forall} \rho(v \cdot n) dA \overset{\text{def}}{=} \begin{cases} Q_m^{\text{in}} & \text{Injection} \\ -Q_m^{\text{wf}} & \text{Ejection} \end{cases} \tag{1}$$

where Q_m^{in} is assumed to be measured by mass flowmeters and Q_m^{wf} can only be predicted by the model. In the middle of the boundaries of the tubing mass is conserved and only changes of pressure, temperature and volume need to be considered. In this conditions assume that a Lagrangean frame is attached to a control volume $\forall_i$ which encloses every fluid i-stage. Between injection and ejection, the total i-stage mass functional $m_i[t]$ remains invariant. We may write then $\frac{Dm_i}{Dt} = 0$. Nevertheless their fluid properties: density ρ, volume $\forall$, average temperature T_{avg} and pressure P_{avg} may change along the pipeline. For this reason a thermodynamic equation of state (EOS) should be used to predict these properties as function of the local wellbore temperature and pressures. In the case of black-oil simulations, correlations are available, see Table 1.

Table 1. Some sources for fluid properties

Fluid	Property	Method	Source
Nat. gas	Compressibility	Correlation	[2]
Nat. gas	Volume factor	Correlation	[2]
HCl	Viscosity	Table	[16]
N_2	Viscosity	Sutherland	[22]
Foam	Viscosity	Reidenbach	[19]

Proposition 1. *Each fluid block contained in a control volume $\forall_i$ restrained by dissipation, body forces and transfer of momentum as in Figure 1 has a momentum equation expressed by*

$$\frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial x} - \frac{1}{\rho}\frac{\partial P}{\partial x} - g\frac{\partial z}{\partial x} - \frac{fv|v|}{2D}. \tag{2}$$

Proof. Consider a control volume $\forall_i$ along a flowline of length ds affected by a body force with associated storage functional $S(x(t)) = W_f = \int_{\forall} \rho f \, d\forall$ for a supply-rate density $r(x(t)) = \rho f$ and supply-rate functional $R[r, \mathrm{v}, A] = -\int_A \rho f \; \mathrm{v} \cdot \mathrm{n} \; dA$. Such body force restrains the associated momentum of $\forall_i$ at the i-stage with a force exerted to the interfaces as pressure given by $f = P/\rho$ and therefore $F_p = \int_A P \; \mathrm{v} \cdot \mathrm{n} \; dA$ and $W_f = \int_{\forall} P \, d\forall$. By conservation of momentum $\frac{DM}{Dt} = F_p + F_\sigma$ *i.e.*

$$\frac{D}{Dt}\int_{\forall} \rho v \, d\forall = \int_A P \, \mathrm{v} \cdot \mathrm{n} \, dA - \int_A (P + \frac{\partial P}{\partial s}) \, ds \; \mathrm{v} \cdot \mathrm{n} \, dA + \int_A \rho\sigma \; \mathrm{v} \cdot \mathrm{n} \, dA$$

otherwise written (with evident intermediate steps) as

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial x} - \rho g \frac{\partial z}{\partial x} - \rho \sigma v \tag{3}$$

where the overall internal dissipating (frictional) forces are assumed to be collected on the term $\sigma \stackrel{\text{def}}{=} f|v|/2D$. With respect to the Eulerian frame, Eq. (3) reduces to the more familiar momentum equation (2). Finally, express $\partial z/\partial x \stackrel{\text{def}}{=} h\cos(\theta) = h\sin(\psi)$ for each staggered fluid section, see Table 2 for Nomenclature.

3 A singular perturbation viewpoint for reduction of two-phase fluid equations

The simplest theory for two-phase flow is the theory of homogeneous flow, see e.g. [21, 8]. In essence, this theory assumes that such two-phase flow behaves as a pseudofluid obeying the usual one-phase equations (namely, mass, momentum and energy conservation) with *weighted averaged properties* such as velocity, density, temperature and viscosity. Throughout the paper we consider the mean density $\rho_m = \rho_L \alpha_L + \rho_G(1-\alpha_L)$, $0 \le \alpha_L \le 1$ and mean viscosity (Cicchitti, [8]) $\mu_m = (1-x)\mu_L + x\mu_G, 0 \le x \le 1$. Since in the oil industry the english unit system is still the most common unit system, in terms of this unit system we write Eqs. (1) and (2) as follows

$$\frac{\partial \rho_m}{\partial t} = -\rho_m \frac{\partial v}{\partial x} - v\frac{\partial \rho_m}{\partial x} \tag{4}$$

$$\epsilon \frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial x} - \frac{1}{\rho_m}\frac{\partial P}{\partial x} - \frac{P}{\rho_m A}\tau_w - \frac{\rho_m g}{g_c}\cos\theta, \tag{5}$$

with enthalpy h and the *perturbation parameter* $\epsilon \approx 1$ to be used in the proof of Prop. 2. The well known fact in control theory that the theory of singular perturbations is very

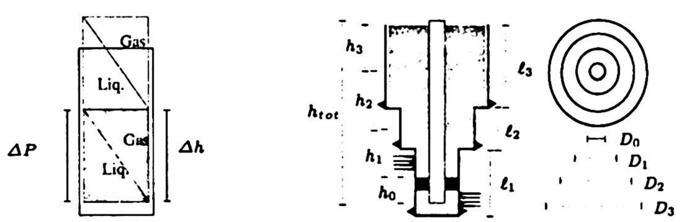

Fig. 2. Conceptual illustration of staggered two-phase blocks and Telescoped tubing

useful for model reduction will be used in Prop. 2 in order to justify a reduced order model for two-phase homogeneous fluids.

3.1 Considerations for incompressible fluids

By defining a control volume moving with the section of fluid injected, the pressure drop including friction effects results in $\Delta P_i = \frac{g}{g_c}\rho h \sin(\frac{\psi\pi}{180}) - \frac{2Ff\rho v^2 h}{g_c D_{tub}}$ with properties obtained from Table 1.

3.2 Considerations for foams

A foam is a dispersion of gas in a solution such that the liquid is the continuous phase and the gas is the discontinuous phase [4]. Foam is used frequently as a diversion fluid for acidizing treatments because is serves for effective placement of the acid. Thus, as the foam stage is traveling along the pipeline its control volume $\forall_{foam}$ is being reduced and the overall contribution to the bottomhole pressure is changed accordingly. Foams are compressible fluids with a rather complex rheological behavior, see [4, 19]. The uncertainty added by the inclusion of such complex fluid increases the difficulty on the estimation of the bottomhole pressure. The rheological properties of foam are dependent of concentration of the surface active agent and temperature. In particular, the apparent viscosity of the foam is usually higher than of each of its constituents and it may decrease with increasing shear rate. It is mostly agreed that foam behaves as a non-newtonian pseudoplastic fluid [4]. In this work we use the *Herschel-Bulkley model* defined by the equation $\tau_{rx} - \tau_y = \mu(-\frac{dv}{dr})^{\frac{1}{m}}$, where τ is the shear stress and the yield stress τ_y is obtained from a flow curve and m and viscosity μ are determined by the slope and intercept of the curve $-\frac{dv}{dr}$ versus $(\tau_{rx} - \tau_y)$. For further information on this model and its parameter adjustment see [10].

3.3 Pressure drop model

In this subsection we formalize the procedure presented by [6] as an interesting particular case of ***model reduction of fluidic two-phase systems*** based on singular perturbations arguments, see e.g. [11]. One distinctive difference of our approach for the calculation of the foam presure drop with the iterative method of [6] lies on the use of an optimization algorithm running in real-time in order to satisfy exactly the momentum equation implied on the calculation of the pressure drop.

Proposition 2 (Two-phase pressure drop model). *Assume that the variation of fluid velocity v is sufficiently lower than the mass flow variation. Then there exist an integral invariant manifold parametrically described as* $\mathcal{M}(P_2, P_1, h_f) = 0$ *where*

$$\mathcal{M}(P_2, P_1, h_f) \stackrel{def}{=} \int_{P_1}^{P_2} H(p)dP - K\frac{h_f}{D_{tub}} \tag{6}$$

such that given P_1 *the pressure* $P_2(P_1, h_f)$ *can be determined as a function of the pipeline height* h_f *by the optimization problem posed as follows*

$$\begin{cases} \textit{Minimize} & \mathcal{M}^2(P_2, P_1, h_f) \\ \textit{Restricted to} & h_i \geq 0, \quad \Delta P_i \geq 0, \quad \forall_i \geq 0 \end{cases} \tag{7}$$

Proof. The proof is appended at the end of the paper.

Our singular perturbation interpretation seems to be original. In the work oriented to field application by [6], this problem is implemented with an iterative method running along the temporal evolution of the estimation, resulting a poor calculation of P_2. This distinguishes our work where the nonlinear optimization problem determines P_2 and h_f simultaneously. Furthermore since the volume $\forall_{foam}$ can be found from $h_f = \forall_{foam}/A_p$, the problem simplifes to finding a root P_2 to equation $\mathcal{M}(P_2, P_1, h_f) = 0$. We used a modified van Wijngaarden-Dekker-Brent's method as the root-finding algorithm, see e.g. [17]. Notice that this approach is independent of the rheologic model (namely, Power law or Herschel-Bulckley) used to parametrize F_f and μ_{foam}.

4 Volume accommodation inside the wellbore

The transient model for the acid stimulation process reduces to a set of control volumes $\{\forall_0, \forall_1, \ldots \forall_n\}$ containing each of the fluids sequentially introduced in the borehole,–according to the stimulation schedule–, with $\forall_0$ the initial fluid inside the borehole before the stimulation and $\forall_1, \ldots \forall_n$ the subsequent injection fluids.
We assume that real-time measurements of the surface pressure P_{ws} and mass flows Q_m^L Q_m^G are available during all the stimulation. Thus the injected volumes can be determined accurately by integration of the mass flow $\frac{dm_i}{dt} = Q_m|_{in} - Q_m|_{out}$ since $m_i = \rho\forall_i$ and $Q_m = \rho q$, (with q the volume flow), this mass conservation equation simplifies trivially to $\frac{d\forall_i}{dt} = q_{in} - q_{out}$, $i = 1, \ldots, n$ and this calculation is performed sequentially to all the fluids injected to the wellbore.

4.1 Estimation of P_{wfi} from the production tubing pressure

Consider Fig. 1 again. Since the total volume $\forall_{tot}$ of the pipe is limited by its physical dimensions, the total sum of the volumes introduced in the pipe, the height and pressures must satisfy

$$\sum_{i=1}^{n} h_i = h_{tot}, \qquad \sum_{i=1}^{n} P_i = P_{wfi}, \qquad \sum_{i=1}^{n} \forall_i = \forall_{tot} \tag{8}$$

All the remaining fluids exceeding the total volume $\forall_{tot}$ are incorporated to the formation at the lowest extreme of the pipe. The total volume $\forall_{tot}$ determines (along with the pipe inclination and transversal area) the maximum vertical length h_{tot} reached by the fluid column during injection and such value assist in determining the bottomhole pressure during injection P_{wfi}.
The additional difficulty comes when some of the fluids are compressible like foam. As the foam is injected along the pipe, their volume $\forall_{foam}$ is decreasing and satisfaction Eq. (8) obligues to modify the whole calculation of P_{wfi} at every time during the injection. While the properties of foam have been studied for a while [4, 19], it is evident that this is an important source of uncertainty during all the calculations. At the begining of the estimulation operations the wellbore is filled with a mixture of residual fluids.

Sometimes this residual fluids are the remains of the pre-test operations performed in order to provide an estimate of the permeability, porosity and initial skin. As the stimulation operation begins, these fluids are displaced following the order prescribed in the *stimulation schedule*. Depending on the type of estimulations performed, most frequently this injection is performed from the production line but sometimes this injection is performed from the annular space. In any case, the appropriate height of each fluid volume must be accounted for every pipeline segment diameter.

4.2 Estimation of P_{wfl} from the annulus surface pressure

Frequently by economy reasons tubing and annulus are used for production of two different zones of a reservoir [8]. In the situation when the stimulation fluids are injected by the annulus (instead of the production tubing) it is necessary to consider in the model the surface reduction of the annular space in the wellbore for telescoped pipelines, see Figure 2. The procedure for estimation of bottomhole pressure in this case consists of performing the previous procedure for constant transversal area iteratively for each segment ℓ_i of equal transversal area of the pipeline and transfering the excedent volumes to the next segment ℓ_{i-1}.

5 Conclusions

From the viewpoint of state estimators (*observers*), the nonlinear model presented in this paper serves as an open-loop estimator. Clearly the prediction of the bottomhole pressure during injection is suceptible of improvement by the appropriate application of *closed-loop state estimators* in order to predict the remotion of skin despite the modeling uncertainty associated to complex fluids like gelled/emulsified acids and foams.

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Table 2. Nomenclature

q Volume flow $[Bl/D]$	μ Viscosity $[cp]$
s skin factor, $-5 < s < \infty$	x spatial coordinate
α_L liquid volume fraction, $0 < \alpha_L < 1$	x gas mass fraction, $0 < \mathrm{x} < 1$

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Proof of Proposition 2

Proof. When the velocity is singularly perturbed by the ϵ-parameter in Eq. (5), we obtain $-\epsilon \frac{\partial v_f}{\partial t} = v_f \frac{\partial v}{\partial x} + \frac{1}{\rho_m} \frac{\partial P}{\partial x} + \rho_m g \cos\theta + \frac{f}{2D} v^2 \approx 0$. We are interested in characterizing the integrant invariant manifold as $\epsilon \to 0$, expressed in parametrized form as $\mathcal{M}(P_2, P_1, h_f) = 0$. In terms of differentials we may express the latter equation by

$v_f dv_f + \frac{1}{\rho} dP - g dz + \frac{f}{2D} v_f^2 dx = 0$ where $dz = -\sin(\theta) dx$. Since $Q_m = \rho_m v_f A_p$ after substitution it yields $v_f = \frac{Q_m}{\rho_f A} = Q_m \forall_f / A = \frac{Q_m}{A}\left(\frac{a+bP}{P}\right)$ such that $dv_f = -\frac{Q_m a}{AP^2} dP$. We may write otherwise

$$\left[\frac{Q_m a}{g_c A}\frac{(a+bP)}{P}\left(-\frac{Q_m a}{AP^2}dP\right)\right] - \frac{g}{g_c} sen\theta d\ell + \frac{(a+bP)}{P}dP + \frac{2F_f}{Dg_c}\left[\frac{Q_m a}{g_c A}\frac{(a+bP)}{P}\right]^2 d\ell = 0$$

which can be integrated after assuming a drag coefficient independent of x. Let us define $\alpha_0 \stackrel{\text{def}}{=} -(Q_m a)^2$, $\alpha_1 \stackrel{\text{def}}{=} -Q_m^2 ab$, $\alpha_2 \stackrel{\text{def}}{=} a g_c A^2$, $\alpha_3 \stackrel{\text{def}}{=} b g_c A^2$, $\beta_1 \stackrel{\text{def}}{=} -2(Q_m a)^2 F_f$, $\beta_2 \stackrel{\text{def}}{=} -4abQ_m^2 F_f$ and $\Delta \stackrel{\text{def}}{=} D_{\text{tub}} g A_p^2 \sin(\frac{\psi\pi}{180}) - 2F_f(bQ_m)^2$. After ordering it yields

$$H(p) = \frac{\alpha_3 p^3 + \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\Delta p^3 + \beta_3 p^2 + \beta_1 p}$$

where ψ is the tubing inclination angle. $H(p)$ can be factorized as

$$H(p) = \frac{\frac{\alpha_3}{\Delta}p^3 + \frac{\alpha_3}{\Delta}p^2 + \frac{\alpha_1}{\Delta}p + \frac{\alpha_0}{\Delta}}{p(p^2 + \frac{\beta_3}{\Delta}p + \frac{\beta_1}{\Delta})} = \frac{\frac{\alpha_3}{\Delta}p^3 + \frac{\alpha_3}{\Delta}p^2 + \frac{\alpha_1}{\Delta}p + \frac{\alpha_0}{\Delta}}{p(p-\lambda_1)(p-\lambda_2)}.$$

In terms of the determinant $\Delta_{det} \stackrel{\text{def}}{=} \left(\frac{\beta_2}{2\Delta}\right)^2 - \frac{\beta_1}{\Delta}$, there may exist two solutions.

If $\Delta_{det} > 0$ the roots $\lambda_{1,2} = \frac{-\beta_2}{2\Delta} \pm \sqrt{\left(\frac{\beta_2}{2\Delta}\right)^2 - \frac{\beta_1}{\Delta}}$ allows us to expand $H(p)$ in partial fracctions as

$$H(p) = \frac{\alpha_3}{\Delta} + \frac{\alpha_0}{\lambda_1\lambda_2\Delta}\frac{1}{P} + \frac{\Xi}{P-\lambda_1} + \frac{\Upsilon}{P-\lambda_2}$$

where $\Upsilon \stackrel{\text{def}}{=} \frac{\omega_1\lambda_2^2 + \omega_2\lambda_2 + (\alpha_0/\Delta)}{\lambda_2(\lambda_2-\lambda_1)}$, $\omega_2 \stackrel{\text{def}}{=} \frac{\alpha_1\Delta - \alpha_3\beta_1}{\Delta^2}$, $\Xi \stackrel{\text{def}}{=} \frac{\omega_1\lambda_1^2 + \omega_2\lambda_1 + (\alpha_0/\Delta)}{\lambda_1(\lambda_1-\lambda_2)}$ and $\omega_1 \stackrel{\text{def}}{=} \frac{\alpha_2\Delta - \alpha_3\beta_2}{\Delta^2}$. Let $K \stackrel{\text{def}}{=} \frac{\Delta}{\alpha_3}$, then the defined integral as a function of pressure is written as

$$\frac{1}{K}\int_{P_1}^{P_2} H(p)dP = P_2 - P_1 + N_1 \log\left(\frac{P_2}{P_1}\right) + N_2 \log\left(\frac{P_2-\lambda_1}{P_1-\lambda_1}\right) + N_3 \log\left(\frac{P_2-\lambda_2}{P_1-\lambda_2}\right)$$

where $N_1 \stackrel{\text{def}}{=} \frac{\alpha_0}{\alpha_1\alpha_2\alpha_3}$, $N_2 \stackrel{\text{def}}{=} \frac{\Delta}{\alpha_3}\Xi$ and $N_3 \stackrel{\text{def}}{=} \frac{\Delta}{\alpha_3}\Upsilon$ such that one may define the integral in Eq. (6) such that $\mathcal{M}(P_2, P_1, h_f) = 0$ i.e. defines an invariant manifold.

When $\Delta_{det} < 0$ the roots $\lambda_{1,2} = \frac{-\beta_2}{2\Delta} \pm \sqrt{\frac{\beta_1}{\Delta} - \left(\frac{\beta_2}{2\Delta}\right)^2}$ allows us to expand $H(p)$ in partial fracctions as

$$H(p) = \frac{\alpha_3}{\Delta} + \frac{\alpha_0}{\lambda_1\lambda_2\Delta}\frac{1}{P} + \frac{AP+B}{(P^2 - \alpha_2 P + \alpha_1)}$$

The integrals implied in the last expansion are of the type $\int \frac{dx}{ax^2+bx+c}$, $\int \frac{x\,dx}{ax^2+bx+c}$ with standard solutions. With similar arguments an equivalent expression for the integral of $H(p)$ is found.